Kinetic grain model for sandpiles

A. Malthe-Sørenssen
Department of Physics, University of Oslo, Box 1048 Blindern, N-0316 Oslo, Norway
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A numerical model for sandpiles is introduced, based on the motion of a single grain and its interaction with a pile instead of the local stability criterion used in traditional models. The model sandpile reaches a stationary state where the energy dissipation events are power law distributed, consistent with the hypothesis of self-organized criticality. The distribution of particles dropping off the pile fits a stretched-exponential distribution as observed in inertia dominated experiments. Both distributions exhibit a finite-size data collapse, indicating that the activity is restricted to a thin zone along the surface. [S1063-651X(96)00806-9]

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I. INTRODUCTION

Sandpiles have been proposed as the canonical example of slowly driven dissipative systems. As grains of sand are added to a pile, the system is driven to a stationary, critical state, where the addition of single grains leads to avalanches of all sizes. In model sandpiles the events are power law distributed, analogous to the behavior near the critical point in equilibrium systems. However, sandpiles organize into a critical state without any explicit tuning of parameters. The notion of self-organized criticality (SOC) [1,2] was proposed to describe this inherent tuning to criticality and describe phenomena as diverse as earthquakes [3], evolution [4], and front dynamics [5].

The dynamics of traditional sandpile models have been based on local stability criteria within the pile [1,2,6–10]. In the cellular automaton model of Bak, Tang, and Wiesenfeld [1,2] grains of sand were added at random positions on a discrete lattice. If the local slope exceeded a threshold value, a fixed number of grains was redistributed to the neighboring positions. The redistribution process was iterated until all local slopes were below the threshold. In more than one dimension this model produced power law probability distributions of the avalanche sizes, measured as the number of redistributions. Even though the model has been modified to include experimental observations such as inertia effects [9] and varying local slopes [10], it is still essentially the local slope that determines whether an avalanche will occur. However, a characteristic of surface flow in granular materials is that the pile is both the medium for and a result of the flow. A strong coupling between flow, dissipation, and the structure of the pile should therefore be included. Recently, Bouchaud *et al.* [11] introduced a model with a coupling between a field of moving grains and the pile surface, though the dynamics of the model still depended on the local slope of the pile.

In this article we present a model for a pile of sand that considers the motion of individual grains. A local stability criterion is not assumed, but will emerge as the result of the frictional interactions. Ideally, the model should be based on molecular-dynamics-type simulations of intergrain contacts, but since we here are interested in the scaling limits and need to study large systems for long times, the large amount of CPU time needed rules out this approach. Thus we apply a cellular model, in which grains are dropped above one end of the pile and gain kinetic energy while bouncing down, transferring energy to particles in the pile. The model displays many of the visual features observed in real (one-dimensional) piles of rice [12]. The energy dissipation is power law distributed, with a finite-size cutoff that scales with the system size, indicating a thin, active zone along the surface.

II. DESCRIPTION OF THE MODEL

In the one-dimensional model, grains are represented as square particles placed at integer positions i. The pile is defined as a series of heights $h(i)$, the number of square particles stacked at position i. A closed boundary condition is applied at position 0, representing a sidewall, and the system is open at position L, representing the outlet. In each cycle a grain is added next to the wall at a height h_0 above the top of the pile. The particle slides down the pile obeying simple rules: It moves down until the pile is directly below it $[y=h(i)+1]$, gaining one unit of kinetic energy for each site it moves down. It then collides with the pile particle [at $h(i)$], transferring some energy e_t to this particle. If it still has a kinetic energy larger than zero, it continues horizontally in the direction of motion, repeating the process above. Otherwise it is left a stationary part of the pile. A particle moves until it loses all its kinetic energy in a local minimum, or escapes off the open boundary. The process is repeated for all particles that have received collisional energy in sequence of *increasing* height, in order to avoid possible ambiguities in grain placement and collision sequences. A newly triggered particle is moved as if it came from above with the energy gained in the collision, though in order to start moving the particle has to overcome a trial collision with the particle below it. If it loses all its energy in this first collision, this energy is absorbed by the pile, reflecting the stability of the pile or static friction between the grains. No further grains are triggered by this energy. If a particle in the pile does not have a grain immediately below it, it will also start falling. When all particles have stopped, a new grain is dropped. Figure 1 illustrates the basic principles of the model.

The parameters of this model are the system size L, the height above the pile h_0, that is, the initial energy when the

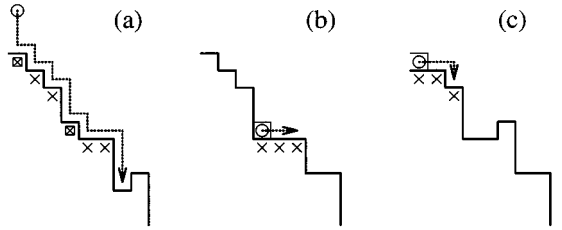

FIG. 1. Examples to illustrate the collision rules. (a) A particle is dropped three units above the top of the pile and stops in a local minimum. The × shows where it has tried to transfer energy. The particles marked with □ surpass the static friction and start moving. (b) The first of the triggered particles stops at the same height; no new particles are initiated. (c) The second triggered particle moves down one unit.

particle first hits the pile, and e_t, the transferred energy, given by a random distribution. Here e_t is uniformly distributed over a range of possible outcomes $\mathcal{R}$. For example, for the distribution $\mathcal{R}=\{0,1,2\}$, the possible e_t values are 0, 1, and 2 with probability 1/3. We will call a choice of transfer distribution ($\mathcal{R}$) a friction rule.

We measure the response of the pile to the addition of a single grain in terms of its avalanches. Adding a grain adds potential energy to the system, which is dissipated through friction as the grain slides down the pile. An avalanche is the size of the energy dissipation following the addition of a grain. This is measured as the height moved downward by all the particles during an avalanche, the loss in potential energy, minus the kinetic energy transferred out of the system. (Each particle may still have some energy when it leaves the system.) We term this quantity s, the avalanche size. The number of grains transferred out of the system during an avalanche is d, the drop number.

III. RESULTS

A few simple examples of friction rules illustrates the model. If the model does not dissipate any energy, that is, $e_t=0$, a single grain will fall all the way through, gaining energy equal to the difference in height of the start and the outlet. The pile will not be modified. Allowing the grain to dissipate 1 on each step $e_t=1$, a single grain added with an initial height h_0 greater than 1 will also run all the way through, but its energy will not increase. The pile will still not be modified. Applying a deterministic model $e_t=E$ from a flat pile causes the pile to build up indefinitely, unless the initial energy scales with the system size. But a stationary configuration of the pile exists for all choices of E, in which a single grain runs through the whole system without gaining energy. To get an interesting interplay between the pile and the particles, randomness must be added to the energy transfer. We will now pursue such models.

For the simplest case $\mathcal{R}=\{0,1\}$, the dynamics becomes more complicated. The model displays a complex, changing surface, similar in appearance to a pile of sand. The pile builds up until it reaches a stationary state, where the addition of a grain leads to redistributions of grains along the surface. Examples of the surface shapes and the redistributions are shown in Fig. 2 for $\mathcal{R}=\{0,1,2,3\}$. A time sequence

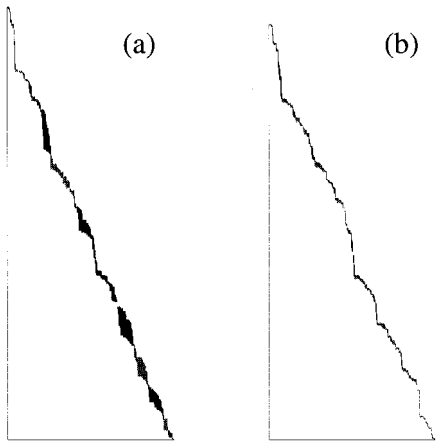

FIG. 2. Two successive pile profiles are shown in the same figure. The black regions represent grains that have been removed in an avalanche, the gray areas denote regions of added grains. A single avalanche therefore consists of transport along the whole surface. The pictures are for $L=200$ and $\mathcal{R}=\{0,1,2,3,4\}$; (a) $s=0.43s_{\max}$. (b) $s=0.02s_{\max}$.

of the energy dissipation s within the pile is shown in Fig. 3. The sequence has structure over many scales, but does not display any clear periodicity. To analyze the process we study the probability density $P(s,L)$ for an avalanche of size s in a system of length L. ($P(s,L)ds$ is the probability for an avalanche in the range $[s,s+ds]$.) Figure 4(a) shows $P(s,L)$ for a $\mathcal{R}=\{0,1\}$ model for different L values. It behaves as a power law $P(s,L)\sim s^{-\alpha}$ for small values of s, with $\alpha=1.55\pm0.05$ for large L, but deviates for large s. The deviations scale with system size as shown by the data collapse in the finite-size scaling plot in Fig. 4(b). Thus $P(s,L)$ can be expressed as

$$P(s,L)=L^{-\beta}g(sL^{-\nu}) \quad (s,L\gg 1). \tag{1}$$

If we also incorporate the small-s behavior, we propose

$$P(s,L)=\begin{cases} 0, & s<s_0 \\ f(s), & s_0<s<s_1 \\ L^{-\beta}g(sL^{-\nu}), & s,L\gg 1, \end{cases} \tag{2}$$

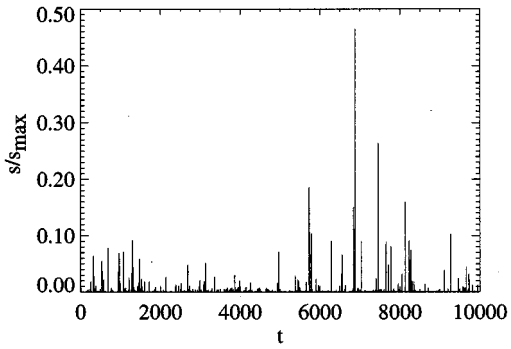

FIG. 3. Time sequence of energy dissipation events for a system with $\mathcal{R}=\{0,1\}$ and $L=200$. The sequence shows structure on several scales, though no clear periodicity is evident. Only a small portion of the total data set (10^7 events) is shown and the events are scaled with the largest avalanche in the whole simulation.

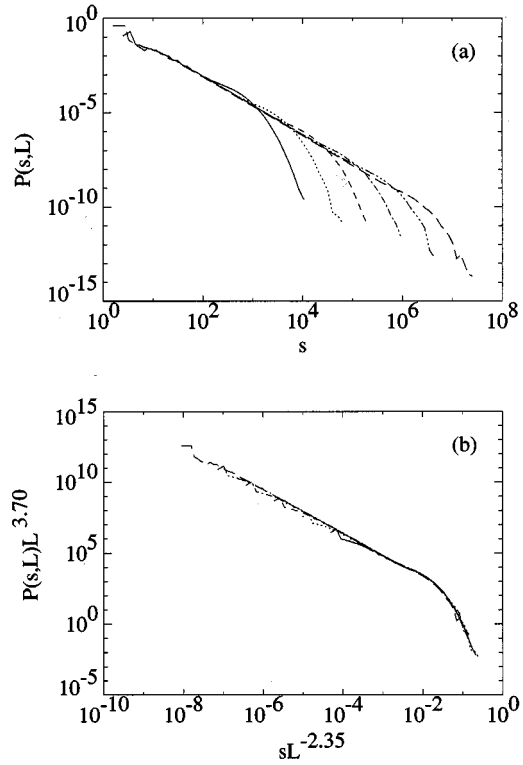

FIG. 4. (a) The avalanche distribution function $P(s,L)$ is plotted for $\mathcal{R}=\{0,1\}$ for $L=100,200,400,800,1600$, and 3200. The data are for 10^7 avalanches after a transient time of 10^7 avalanches. (b) A finite-size scaling plot: $P(s,L)$ is plotted with rescaled axes to show a functional dependence according to $P(s,L)=L^{-\beta}g(sL^{-\nu})$. The exponents $\beta=3.7$ and $\nu=2.35$ give the best possible data collapse for large L and s. The exponent in the straight line section is $\alpha=-1.55\pm0.05$.

where $g(x)$ is a power law $x^{-\alpha}$ with a cutoff for large x. Since the smallest avalanche is h_0, s_0 is identified as h_0. The plots in Fig. 4 show that all distributions deviate from the power law $s^{-\alpha}$ for $s<s_1$, where $s_1\simeq 20$. This behavior and the value of s_1 are the same for all models ($L\gg 1$) and independent of L, since only the large avalanches detect the size of the system. It was therefore not possible to obtain a simultaneous data collapse of both the large and small avalanches. This does, however, represent the complete scaling behavior and a multifractal analysis will not give more information.

For larger avalanches, the scaling exponents $\beta=3.7$ and $\nu=2.35$ were found to give the best data collapse (for large L). Several scaling relations are imposed on the system. The average dissipated energy must on average be equal to the input energy $E_{\rm in}$. Since the variation with system size L in the angle of repose θ is small and the height of the pile is $h=L\tan(\theta)$, the input energy is $E_{\rm in}=L\tan(\theta)+h_0$. $E_{\rm in}$ therefore scales with L since $h_0\ll L$ and is independent on L. This implies that $2\nu-\beta=1$ [12]. Normalization of the probability density requires that

$$\int_0^\infty P(s,L)ds=1=\int_{s_0}^{s_1}f(s)ds+\int_{s1}^{\infty}L^{-\beta}g(sL^{-\nu})ds, \tag{3}$$

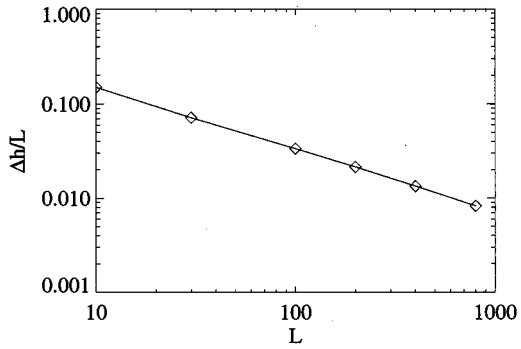

FIG. 5. Width of the pile $\triangle h/L$ as a function of the system size L. The curve is consistent with a power-law behavior $\triangle h\sim L^{\chi}$, where $\chi=0.35\pm0.02$.

where the first term is a number and the second term is dominated by the lower limit, since $\alpha<2$. The resulting scaling relation is $\nu-\beta-\nu(1-\alpha)=0$. We observe that the found ν and β gives $\alpha=1.57$, which is consistent with the measured value of α.

The energy of the largest avalanche $E^{\star}\sim L^{\nu}$ corresponds to the energy of the particles in a zone of thickness ξ along the length of the surface. The whole pile has a potential energy that scales as L^3 and a zone of thickness ξ has a potential energy scaling as ξL^2. The measured value of ν is therefore consistent with a zone of width $\xi\sim L^{\chi}$, $\chi=0.35$. The width of the active zone was also measured directly in the simulations. For a pile of size L, $\triangle h(i;L)$ is the standard deviation of the height at position $i=1,2,\ldots,L$ and $\triangle h$ is the average over i. Figure 5 shows the scaling of $\triangle h$ with L. The plot suggests a power-law relation $\triangle h\sim L^{\chi}$, with $\chi=0.35\pm0.02$. We also observe that the scaling argument above shows that for $\alpha=3/2$, $\beta=3$ and $\nu=2$. This produces a significantly poorer data collapse and would also imply a zone of constant width, inconsistent with the observed behavior.

Though the large avalanches are rare, their contribution dominates the energy dissipation in the system, as is the case for any distribution with an exponent less than 2. We also observe that the distributions $P(s,L)$ display a large event ''bump'': For large avalanches the curve turns slightly above the straight line of the power-law behavior before falling off. This is interpreted as a finite-size effect and is a part of the finite-size cutoff that scales with the system size.

Only some of the avalanches result in grains dropping off the edge of the system. Figure 6 shows the probability of such drop events as a function of system size L. The time sequence of drop events d, shown in Fig. 7, also shows structure over several scales and no evident periodicity. The probability distribution for drops $P(d,L)$ is not a power law over any range, as evident from Fig. 8(a). Notice that this is a conditional probability density, given that a drop occurs. Since the conditional probabilities are normalized, the scaling relation $2\nu'-\beta'=0$, which follows since $\langle d\rangle\sim L^0$, is not valid. But the relation $\beta'=\nu'$ holds as a result of normalization since the data collapse is obtained both in the large and small value limit and the scaling function has a finite integral. The larger drops are consistent with a stretched-exponential distribution

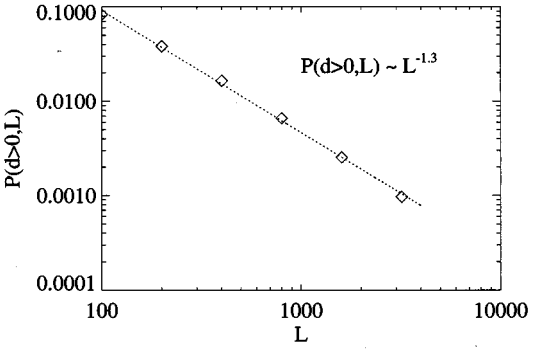

FIG. 6. Probability of a drop event $P(d>0,L)$ as a function of system size L. The probability decreases with system size. Most events in the system are therefore internal events that do not reach the edge of the system. The curve represents a fit $P(d>0,L)\sim L^{\gamma}$ to the points, with $\gamma=-1.3\pm0.05$.

$$P(d,L)=AL^{-\beta'}\exp[-a(d/L^{\nu'})^{\gamma}] \quad \text{for} \quad d,L\gg 1 \quad (4)$$

and scale with the system size as shown in Fig. 8(c). This is in very good agreement with a recent reanalysis [13] of experimental results [14,15] by Feder showing that stretched-exponential distributions give an excellent fit. The scaling collapse in Fig. 8(b) indicates that the largest drop scales as $L^{\chi+1}$, consistent with the dismantling of a zone of width ξ along the surface. The drop data are also consistent with a pure exponential distribution, though the fit is slightly better for the stretched-exponential distribution.

Changing the other parameters of the model does not have significant effect on the statistics. The behavior of the model is not affected by changes in the initial energy of the grains h_0. The same statistics are also observed if the amount of energy transferred out of the system is not subtracted. Though increasing the friction for dissipation rules of the form $\mathcal{R}=\{0,1,\ldots,k\}$, decreases the influence of the large event bump, the exponents of the distribution and the finite-size scaling data collapses remain the same. However, for higher values of k, the pile built up a steeper slope and the fluctuations in the pile increased. Other types of friction rules did also not change the behavior significantly.

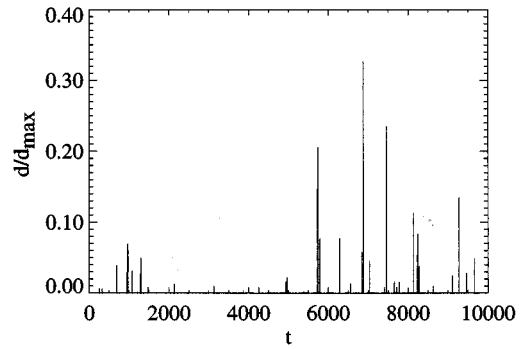

FIG. 7. Time sequence of drop events d for a system with $\mathcal{R}=\{0,1\}$ and $L=200$. The sequence shows structure on several scales, though no clear periodicity is evident. Only a small portion of the total data set is shown and the events are scaled with the largest drop in the whole simulation.

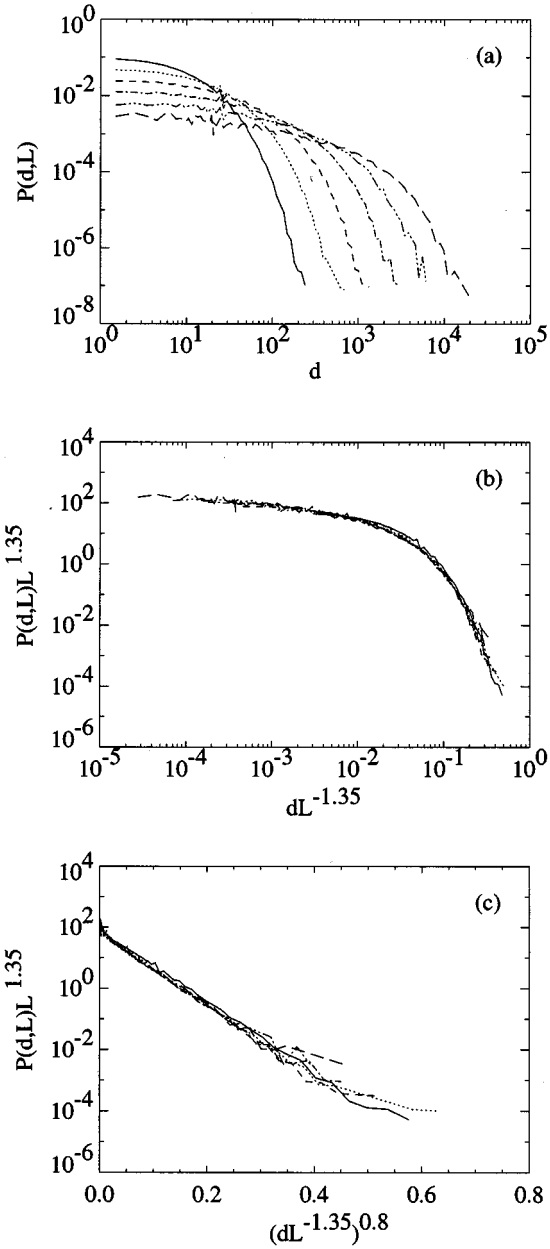

FIG. 8. Finite-size plot of the normalized conditional probability density for a drop d, $P(d,L)=p(d,L|d>0)$, for a model with $\mathcal{R}=\{0,1\}$ and $L=100,200,400,800,1600$, and 3200. The exponents $\beta'=1.35\pm0.05$ and $\nu'=1.35\pm0.05$ produce the best data collapse. The distribution is not a power law in any region (a) and (b), but fits well to a stretched exponential for large d (c).

The value of the exponent α is close to the mean-field values found from the theory of branching processes [16,17]. The model can be approximated by a branching process if grains move down the surface with constant velocity, constant probability to stop, and to knock loose other grains and if changes in the pile surface are ignored. The dissipated energy would then correspond to the length of a branched tree, starting at the top with separate branches for each activated particle [16]. The interesting part in our model is that the system organizes itself into the critical state and that there is a strong coupling both with the pile and the history of the grains, since the probability of a grain stopping de-

pends on its kinetic energy, which is determined by its previous motion.

IV. CONCLUSION

The model represents a simple way to describe the complicated dynamics in a pile of sand. It has the same type of intermittent behavior with scaling, power-law probability distributions as observed in sandpile automata [2,6,7], connects with the mean-field theory, and is consistent with experimental results. It does not contain any explicit details of local stability; the behavior of the pile is determined by kinetic rules for grain-pile interaction. Independent of the details of the grain-pile frictional interactions, the same type of behavior is observed, though the inclination of the pile and the avalanche sizes vary.

We observe a power law for the probability density of avalanches, consistent with the hypothesis of SOC, even though the total energy dissipation is dominated by rare, system-spanning events. However, the drop numbers are consistent with stretched-exponential distributions, indicating a characteristic drop size and in good correspondence with a recent reanalysis of inertia dominated experiments. We obtain a data collapse for both quantities, indicating that the dynamics is restricted to a zone of thickness $\xi \sim L^{0.35}$ along the surface of the pile. This is confirmed by direct measurement of the width ξ as a function of L in the simulations.

According to the hypothesis of SOC, the duration of events should also be power law distributed. The presented model does not have any natural time scale and the avalanche lifetimes therefore cannot be discussed. However, correlations in the sequence of events will be addressed later. The model displays intriguingly complex, critical behavior in one dimension, but a further development of a two-dimensional model would also be of interest, as well as introducing a difference between static and dynamic friction, as indeed is present in real granular materials.

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